Online Exam — Analysis (WBMA012-05)

Tuesday 26 January 2021, 8.30h–11.30h CET (plus 30 minutes for uploading)

University of Groningen

Instructions

1. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
2. If p is the number of marks then the exam grade is $G = 1 + p/10$.
3. Write both your name and student number on the answer sheets!
4. This exam comes in two versions. Both versions consist of six problems of equal difficulty.

   Make version 1 if your student number is odd.

   Make version 2 if your student number is even.

   For example, if your student number is 1277456, which is even, then you have to make version 2.
5. Save your work as a single PDF file and submit it via this dedicated Nestor page.

Version 1: odd student numbers only!

Problem 1 (5 + 5 + 5 = 15 points)

(a) Assume that $B \subset \mathbb{R}$ is nonempty and bounded above. Assume that A is nonempty and $A \subset B$. Show that A is bounded above and $\sup A \leq \sup B$.

(b) Assume that the sets $U, V \subset \mathbb{R}$ are nonempty and bounded above. In addition, assume that $U \cap V$ is nonempty. Use part (a) to show that

$$\sup(U \cap V) \leq \min\{\sup U, \sup V\}.$$

(c) Give an example of sets U and V for which the inequality in part (b) is strict.

Problem 2 (5 + 5 + 5 = 15 points)

Give an example of each of the following, or argue that such a request is impossible:

(a) A sequence (x_n) such that $\lim x_n = 0$ and $x_n = 1$ for infinitely many $n \in \mathbb{N}$.

(b) A divergent sequence (x_n) for which every subsequence (x_{n_k}) converges.

(c) A sequence (x_n) such that $0 \leq x_n \leq 1/n^2$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}(-1)^n x_n$ diverges.

Problem 3 (5 + 5 + 5 = 15 points)

Consider a set $K \subset \mathbb{R}$ that satisfies the following properties:

(i) K is nonempty and compact;

(ii) for all $x \in K$ there exists $\epsilon_x > 0$ such that $K \cap V_{\epsilon_x}(x) = \{x\}$.

Prove the following statements:

(a) The set $A = \{0, 1\}$ satisfies both properties (i) and (ii).

(b) The set $B = [0, 1]$ satisfies property (i), but *not* property (ii).

(c) Any set K that satisfies both properties (i) and (ii) is finite.

Problem 4 (10 + 5 = 15 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and assume that the derivative is bounded, i.e., there exists $M \geq 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Moreover, assume that $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(a) Prove that the function h given by $h(x) = f(g(x))$ is uniformly continuous on $\mathbb{R}$.

(b) Is the function h still uniformly continuous on $\mathbb{R}$ when f does *not* have a bounded derivative? If so, give a proof; otherwise, give a counterexample.

Problem 5 (3 + 6 + 6 = 15 points)

Let $g : [0,1] \to \mathbb{R}$ be a continuous function and consider the sequence (f_n) given by

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = x^n g(x).$$

Prove the following statements:

(a) The sequence (f_n) converges pointwise to $f : [0,1] \to \mathbb{R}$ where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ g(1) & \text{if } x = 1. \end{cases}$$

(b) The convergence $f_n \to f$ is uniform on $[0,b]$ for all $0 < b < 1$.

(c) If the convergence $f_n \to f$ is uniform on $[0,1]$, then $g(1) = 0$.

Problem 6 (3 + 12 = 15 points)

(a) Argue that the function $f(x) = 1/x$ is integrable on $[1,2]$.

(b) Use the partition $P = \{(k+n)/n \,:\, k = 0, \ldots, n\}$ to prove the following inequality:

$$\ln(2) \leq \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} \quad \text{for all} \quad n \in \mathbb{N}.$$

End of version 1 (90 points)

Version 2: even student numbers only!

Problem 1 (5 + 5 + 5 = 15 points)

(a) Assume that $B \subset \mathbb{R}$ is nonempty and bounded below. Assume that A is nonempty and $A \subset B$. Show that A is bounded below and $\inf A \geq \inf B$.

(b) Assume that the sets $U, V \subset \mathbb{R}$ are nonempty and bounded below. In addition, assume that $U \cap V$ is nonempty. Use part (a) to show that

$$\inf(U \cap V) \geq \max\{\inf U, \inf V\}.$$

(c) Give an example of sets U and V for which the inequality in part (b) is strict.

Problem 2 (5 + 5 + 5 = 15 points)

Give an example of each of the following, or argue that such a request is impossible:

(a) A sequence (x_n) such that $\lim x_n = 1$ and $x_n = 0$ for infinitely many $n \in \mathbb{N}$.

(b) A bounded sequence (x_n) for which every subsequence (x_{n_k}) diverges.

(c) A sequence (x_n) such that $0 \leq x_n \leq 1/n^2$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}(-1)^n x_n$ diverges.

Problem 3 (5 + 5 + 5 = 15 points)

Consider a set $K \subset \mathbb{R}$ that satisfies the following properties:

(i) K is nonempty and compact;

(ii) for all $x \in K$ there exists $\epsilon_x > 0$ such that $K \cap V_{\epsilon_x}(x) = \{x\}$.

Prove the following statements:

(a) The set $A = \{0, 1\}$ satisfies both properties (i) and (ii).

(b) The set $B = [0, 1]$ satisfies property (i), but *not* property (ii).

(c) Any set K that satisfies both properties (i) and (ii) is finite.

Problem 4 (10 + 5 = 15 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and assume that the derivative is bounded, i.e., there exists $M \geq 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Moreover, assume that $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(a) Prove that the function h given by $h(x) = f(g(x))$ is uniformly continuous on $\mathbb{R}$.

(b) Is the function h still uniformly continuous on $\mathbb{R}$ when g is *not* uniformly continuous? If so, give a proof; otherwise, give a counterexample.

Problem 5 (3 + 6 + 6 = 15 points)

Let $g : [0,1] \to \mathbb{R}$ be a continuous function and consider the sequence (f_n) given by

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = (1-x)^n g(x).$$

Prove the following statements:

(a) The sequence (f_n) converges pointwise to $f : [0,1] \to \mathbb{R}$ where

$$f(x) = \begin{cases} g(0) & \text{if } x = 0, \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

(b) The convergence $f_n \to f$ is uniform on $[a,1]$ for all $0 < a < 1$.

(c) If the convergence $f_n \to f$ is uniform on $[0,1]$, then $g(0) = 0$.

Problem 6 (3 + 12 = 15 points)

(a) Argue that the function $f(x) = 1/(1+x)$ is integrable on $[0,1]$.

(b) Use the partition $P = \{k/n \,:\, k = 0, \ldots, n\}$ to prove the following inequality:

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \leq \ln(2) \quad \text{for all} \quad n \in \mathbb{N}.$$

End of version 2 (90 points)

Solution of problem 1, version 1 (5 + 5 + 5 = 15 points)

(a) If $a \in A$, then $a \in B$. Since $\sup B$ is an upper bound for B we have that $a \leq \sup B$. Since $a \in A$ is arbitrary, it follows that $\sup B$ is also an upper bound for A.
(3 points)

By definition of least upper bound it then follows that $\sup A \leq \sup B$.
(2 points)

(b) We have that $U \cap V \subset U$. By part (a) it follows that $\sup(U \cap V) \leq \sup U$. By a similar argument it follows that $\sup(U \cap V) \leq \sup V$.
(3 points)

Without loss of generality we may assume that $\sup U \leq \sup V$ (otherwise, just interchange the roles of U and V). Therefore, we obtain

$$\sup(U \cap V) \leq \sup U = \min\{\sup U, \sup V\}.$$

(2 points)

(c) For $U = \{0, 1\}$ and $V = \{0, 2\}$ we have

$$\sup U = 1 \quad \text{and} \quad \sup V = 2,$$

which gives $\min\{\sup U, \sup V\} = 1$. Since $U \cap V = \{0\}$ we have $\sup(U \cap V) = 0$.
(5 points)

Solution of problem 1, version 2 (5 + 5 + 5 = 15 points)

(a) If $a \in A$, then $a \in B$. Since $\inf B$ is a lower bound for B we have that $a \geq \sup B$. Since $a \in A$ is arbitrary, it follows that $\inf B$ is also a lower bound for A.
(3 points)

By definition of greatest lower bound it then follows that $\inf A \geq \inf B$.
(2 points)

(b) We have that $U \cap V \subset U$. By part (a) it follows that $\inf(U \cap V) \geq \inf U$. By a similar argument it follows that $\inf(U \cap V) \geq \inf V$.
(3 points)

Without loss of generality we may assume that $\inf U \leq \inf V$ (otherwise, just interchange the roles of U and V). Therefore, we obtain

$$\inf(U \cap V) \geq \inf V = \max\{\inf U, \inf V\}.$$

(2 points)

(c) For $U = \{-1, 0\}$ and $V = \{-2, 0\}$ we have

$$\inf U = -1 \quad \text{and} \quad \inf V = -2,$$

which gives $\max\{\inf U, \inf V\} = -1$. Since $U \cap V = \{0\}$ we have $\inf(U \cap V) = 0$.
(5 points)

Solution of problem 2, version 1 (5 + 5 + 5 = 15 points)

(a) This request is impossible. Indeed, let $\epsilon = \frac{1}{2}$. Then there exists $N \in \mathbb{N}$ such that $x_n \in V_\epsilon(0) = (-\frac{1}{2}, \frac{1}{2})$ for all $n \geq N$. So there can at most be finitely many $n \in \mathbb{N}$ for which $x_n = 1$. (Note: this argument works with any $0 < \epsilon < 1$.)
(5 points)

Alternative argument. If $x_n = 1$ for infinitely many $n \in \mathbb{N}$, then (x_n) has a subsequence converging to a different limit than the sequence itself. This contradicts the theorem that states that all subsequences of a convergent sequence are convergent and must have the same limit as the sequence itself.

(b) This request is impossible. Indeed, one possible subsequence is obtained by simply taking the sequence itself (by choosing $n_k = k$).
(5 points)

(c) This request is impossible. Indeed, we have that $|(-1)^n x_n| \leq 1/n^2$. Since $\sum_{n=1}^\infty 1/n^2$ is a convergent series, the comparison test implies that the series $\sum_{n=1}^\infty |(-1)^n x_n|$ converges and hence the series $\sum_{n=1}^\infty (-1)^n x_n$ itself converges.
(5 points)

Solution of problem 2, version 2 (5 + 5 + 5 = 15 points)

(a) This request is impossible. Indeed, let $\epsilon = \frac{1}{2}$. Then there exists $N \in \mathbb{N}$ such that $x_n \in V_\epsilon(0) = (\frac{1}{2}, \frac{3}{2})$ for all $n \geq N$. So there can at most be finitely many $n \in \mathbb{N}$ for which $x_n = 0$. (Note: this argument works with any $0 < \epsilon < 1$.)
(5 points)

(b) This request is impossible. Indeed, the Bolzano-Weierstrass Theorem guarantees the existence of at least one convergent subsequence.
(5 points)

(c) This request is impossible. Indeed, we have that $|(-1)^n x_n| \leq 1/n^2$. Since $\sum_{n=1}^\infty 1/n^2$ is a convergent series, the comparison test implies that the series $\sum_{n=1}^\infty |(-1)^n x_n|$ converges and hence the series $\sum_{n=1}^\infty (-1)^n x_n$ itself converges.
(5 points)

Remark. In part (c) one cannot use the Alternating Series Test because that theorem would require that (x_n) is decreasing: $0 \leq x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. This is *not* implied by the given inequality $0 \leq x_n \leq 1/n^2$. Indeed, the sequence $(1, 0, \frac{1}{9}, 0, \frac{1}{25}, 0, \frac{1}{49}, 0, \dots)$ satisfies the given inequality but is not decreasing.

Solution of problem 3, version 1 and 2 (5 + 5 + 5 = 15 points)

(a) The set A is clearly nonempty as it contains two elements.
(1 point)

In addition, the set is compact, since in the lectures it has been shown that finite sets are compact. Hence, the set A satisfies property (i).
(1 point)

For $\epsilon = \frac{1}{2}$ (in fact any $0 < \epsilon \leq 1$ works) we have

$$V_\epsilon(0) \cap A = (-\tfrac{1}{2}, \tfrac{1}{2}) \cap \{0, 1\} = \{0\}.$$

For the point $a = 1$ the reasoning is similar. This shows that the set A satisfies property (ii).
(3 points)

(b) The set B is clearly nonempty as $0 \in B$.
(1 point)

In addition, the set is compact, since in the lectures it has been shown that closed and bounded intervals are compact. Hence, the set B satisfies property (i).
(1 point)

Method 1. Let $\epsilon > 0$ be arbitrary. We have

$$V_\epsilon(0) \cap B = (-\epsilon, \epsilon) \cap [0, 1] = \begin{cases} [0, 1] & \text{if } \epsilon \geq 1 \\ [0, \epsilon) & \text{if } \epsilon < 1 \end{cases}.$$

This implies that for any $\epsilon > 0$ we have $V_\epsilon(0) \cap B \neq \{0\}$, which shows that the set B does not satisfy property (ii).
(3 points)

Method 2. Note that $x = 0$ is a limit point of A. Indeed, the sequence $x_n = 1/n$ lies in B and satisfies $x_n \neq 0$ for all n while $\lim x_n = 0$. By definition of a limit point we then have that for each $\epsilon > 0$ there exists $y \in B$ such that $y \neq 0$ and $y \in B \cap V_\epsilon(0)$. Therefore, the set B does not satisfy property (ii).
(3 points)

(c) The sets $V_{\epsilon_x}(x)$, where $x \in K$, form an open cover for K. Since K is assumed to be compact, it follows that there exist finitely many points $x_1, \ldots, x_n \in K$ such that

$$K \subset V_{\epsilon_{x_1}}(x_1) \cup \cdots \cup V_{\epsilon_{x_n}}(x_n).$$

(3 points)

Since $K \cap V_{\epsilon_{x_i}}(x_i) = \{x_i\}$ for all $i = 1, \ldots, n$ we have that $K \subset \{x_1, \ldots, x_n\}$, which shows that K is a finite set.
(2 points)

Alternative argument. If K is infinite, then the boundedness of K implies the existence of a limit point; this can be shown using a bisection argument as in the proof of the Bolzano-Weierstrass Theorem. The closedness of K would then imply that this limit point is contained in K. This is a direct contradiction with property (ii), which states that no point of K is a limit point.

Solution of problem 4, version 1 and 2 (10 + 5 = 15 points)

(a) If $g(x) \neq g(y)$, then we may assume without loss of generality that $g(x) < g(y)$. By the Mean Value Theorem there exists $c \in (g(x), g(y))$ such that

$$f(g(x)) - f(g(y)) = f'(c)(g(x) - g(y)).$$

(3 points)

Taking absolute values and the boundedness assumption on f' gives

$$|f(g(x)) - f(g(y))| = |f'(c)|\,|g(x) - g(y)| \leq M|g(x) - g(y)|.$$

(1 point)

If $g(x) = g(y)$, then the above inequality trivially holds.
(1 point)

Since g is assumed to be uniformly continuous on $\mathbb{R}$ it follows that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |g(x) - g(y)| < \frac{\epsilon}{M}.$$

(2 points)

Therefore, if $|x - y| < \delta$, then

$$|h(x) - h(y)| = |f(g(x)) - f(g(y))| \leq M|g(x) - g(y)| < M \cdot \frac{\epsilon}{M} = \epsilon.$$

This shows that h is also uniformly continuous on $\mathbb{R}$.
(3 points)

(b) *Version 1.* No, without the assumption that the derivative of f is bounded, the function h need not be uniformly continuous on $\mathbb{R}$. A counter example is given by $f(x) = x^2$ and $g(x) = x$. Clearly, g is uniformly continuous, but f does not have a bounded derivative. The function $h(x) = f(g(x)) = x^2$ is not uniformly continuous on $\mathbb{R}$ as has been shown in the lectures.
(5 points)

Version 2. No, without the assumption that g is uniformly continuous, the function h need not be uniformly continuous on $\mathbb{R}$. A counter example is given by $f(x) = x$ and $g(x) = x^2$. It has been shown in the lectures that g is not uniformly continuous, but f does have a bounded derivative. The function $h(x) = f(g(x)) = x^2$ is not uniformly continuous on $\mathbb{R}$.
(5 points)

Remark. In part (b) a specific example of f and g must be given for which h is not uniformly continuous. It is not possible to state in general that h will not be uniformly continuous because we can find examples for which h in fact is uniformly continuous. For version 1 we could take: $f(x) = x^2$ and $g(x) = 0$ for which $h(x) = 0$ is clearly uniformly continuous on $\mathbb{R}$. For version 2 we could take $f(x) = 0$ and $g(x) = x^2$.

Solution of problem 5, version 1 (3 + 6 + 6 = 15 points)

(a) If $0 \leq x < 1$, then $\lim f_n(x) = \lim x^n g(x) = g(x) \lim x^n = g(x) \cdot 0 = 0$.
(2 points)

If $x = 1$, then $f_n(x) = g(1)$ for all n so that $\lim f_n(x) = g(1)$.
(1 point)

(b) Let $0 < b < 1$ be arbitrary. The function g is continuous on the compact set $[0, b]$ and hence attains a maximum and a minimum. In particular, this implies that g is bounded which means that there exists a constant $M > 0$ such that $|g(x)| \leq M$ for all $x \in [0, b]$.
(2 points)

Therefore, using that $f(x) = 0$ on $[0, b]$, we obtain that

$$\sup_{x\in[0,b]} |f_n(x) - f(x)| = \sup_{x\in[0,b]} |f_n(x)| = \sup_{x\in[0,b]} x^n |g(x)| \leq M \sup_{x\in[0,b]} x^n = M b^n.$$

(2 points)

This implies that

$$\lim \left(\sup_{x\in[0,b]} |f_n(x) - f(x)| \right) = 0,$$

which means that $f_n \to f$ uniformly on $[0, b]$.
(2 points)

(c) Note that each function f_n is continuous since it is a product of continuous functions. If $f_n \to f$ uniformly on $[0, 1]$, then f is continuous as well.
(3 points)

If $g(1) = 0$, then f is identically zero and hence continuous. On the other hand, if $g(1) \neq 0$, then f is not continuous. Indeed, for $x_n = 1 - 1/n$, we have $\lim f(x_n) = 0$, whereas $f(1) = g(1) \neq 0$.
(3 points)

Therefore, we conclude that if $f_n \to f$ uniformly on $[0, 1]$, then $g(1) = 0$.

Remark. For version 2 the arguments are completely analogous to those given above for version 1.

Solution of problem 6, version 1 (3 + 12 = 15 points)

(a) *Method 1.* The function is decreasing and in the lectures it has been shown that decreasing functions are integrable.
(3 points)

Method 2. The function is continuous and in the lectures it has been shown that continous functions are integrable.
(3 points)

(b) Since for $F(x) = \ln(x)$ we have $F'(x) = 1/x$, it follows by the Fundamental Theorem of Calculus that

$$\int_1^2 \frac{1}{x}\,dx = \ln(2) - \ln(1) = \ln(2).$$

(3 points)

Since f is decreasing it follows that

$$M_k := \sup\{f(x) \,:\, x \in [x_{k-1}, x_k]\} = f(x_{k-1}).$$

(3 points)

For the partition $P = \{(k+n)/n \,:\, k = 0, \ldots, n\}$ we thus get the following upper sum

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} \frac{n}{k-1+n}\left(\frac{k+n}{n} - \frac{k-1+n}{n}\right) \\ &= \sum_{k=1}^{n} \frac{1}{k-1+n} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}. \end{aligned}$$

(5 points)

Finally, since $\int_1^2 f \leq U(f,P)$ for any partition P we obtain the desired inequality.
(1 point)

Solution of problem 6, version 2 (3 + 12 = 15 points)

(a) *Method 1.* The function is decreasing and in the lectures it has been shown that decreasing functions are integrable.
(3 points)

Method 2. The function is continuous and in the lectures it has been shown that continous functions are integrable.
(3 points)

(b) Since for $F(x) = \ln(1+x)$ we have $F'(x) = 1/(1+x)$, it follows by the Fundamental Theorem of Calculus that

$$\int_0^1 \frac{1}{1+x}\,dx = \ln(2) - \ln(1) = \ln(2).$$

(3 points)

Since f is decreasing it follows that

$$m_k := \inf\{f(x) \,:\, x \in [x_{k-1}, x_k]\} = f(x_k).$$

(3 points)

For the partition $P = \{k/n \,:\, k = 0, \dots, n\}$ we thus get the following lower sum

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} \frac{1}{1+k/n}\left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}. \end{aligned}$$

(5 points)

Finally, since $L(f,P) \leq \int_0^1 f$ for any partition P we obtain the desired inequality.
(1 point)